Derived algebraic geometry and representation theory II — Tina

Plan: 1) Hecke categories and the main theorem

2) Explanation of notation

a) quasi-coherent sheaves

b) D-modules

c) loop spaces

3) Geometric Langlands conjecture

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1) Hecke categories and the main theorem

G connected reductive algebraic group$/k$ ← char $= 0$

Def^n: The finite Hecke category is

$$\mathcal{H}_G := ~~[illegible]~~ D_{coh}(\mathbf{B}\backslash G/\mathbf{B})$$

cf from yesterday: $\mathbb{H} = \mathbb{C}[B\backslash G/B]$

Def^n: The affine Hecke category is

$$\mathcal{H}_G^{aff} := Coh_{[\mathcal{B}\times\mathcal{B}/G]}(St^u/G)^{\mathbb{G}_m}$$

unipotent gp version of St from yesterday.

Question: What does $\mathcal{H}_G$ have to do with $\mathcal{H}_G^{aff}$? cf. $\mathbb{H}^{fin} \subset \mathbb{H}^{aff}$

Answer: [Ben-Zvi — Nadler]

There is a canonical equivalence of ∞-categories

$$D_{coh}(\mathbf{B}\backslash G/\mathbf{B}) \simeq Coh_{[\mathcal{B}\times\mathcal{B}/G]}(St^u/G)^{S}_{loc}$$

$\boxed{\text{Coherent sheaves}}$: for dg categories

- classical: $\mathrm{Coh}(\mathrm{Spec}\, R) = R\text{-mod}^{f.g.}$

<u>Remark</u>: Yesterday, we used this: $K^G(X) = K_0(D^b\mathrm{Coh}^G(X))$

- for dg categories: X underived stack

$\mathrm{Coh}(X)$ is the full subcategory of $\mathrm{QCoh}(X)$ with bounded and coherent cohomology

<u>Def'n</u> $e: Z \hookrightarrow X$ closed embedding of stacks (underived)

$\mathrm{Coh}_{[Z]}(X) :=$ dg derived category of coherent sheaves $\mathcal{F}$ whose restriction $e^*\mathcal{F}$ is coherent

<u>Remark</u>: X smooth, then $\mathrm{Coh}(X) \simeq \mathrm{Perf}(X)$

locally isomorphic to bounded complex of free $\mathcal{O}_X$-mod of finite type.
= dualizable objects

$\boxed{\text{D-modules}}$: - classically,

<u>Def'n</u> A <u>differential operator</u> on $\mathrm{Spec}\, R =: S$ is $A \in \mathrm{End}(R)$ <u>of degree n</u> satisfying

$$[\dots[[A, f_n], f_{n-1}] \dots f_0] = 0 \quad \forall f_0, \dots, f_n \in R$$

multiplication by $f_n \in R$

Def'n/Remark: $D^{\leq n}(S)$... diff operators of degree $\leq n$ on S

$D^{\leq 0}(S) = R$

$$D(S) := \bigoplus_{n \geq 0} D^{\leq n}(S) =: \text{diff operators on } S = \mathrm{Spec}\, R$$

(filtration!)

Def'n: A dg coherent sheaf is G-equivariant if each term is G-equivariant and differential and act are G-equivariant

notation: $\mathrm{Coh}(X)^G$

Prop: $\mathrm{Coh}(X)^G \simeq \mathrm{Coh}(X/G)$ if X/G is a scheme.

$\boxed{\text{Loop spaces}}$ Recall: $S^1 = * \amalg_{* \amalg *} *$

Def'n: Let X be a derived stack. $\mathcal{L}X := \mathrm{Map}_{dSt}(S^1, X)$

Affinization:

- classical: $\mathrm{Aff}(X) = \mathrm{Spec}\, \mathcal{O}(X)$
- in DAG: $\mathrm{Spec}: \mathrm{cdga}_k^{\leq 0} \longrightarrow dSt$

$\mathcal{O}$ adjoint

$X = \operatorname{colim}_{\mathrm{Spec} A \to X} \mathrm{Spec}\, A$

$\mathcal{O}(X) = \lim_{\mathrm{Spec} A \to X} A$

Example: $\mathrm{Aff}(S^1) = BG_a = \mathrm{Spec}(k \oplus k\underset{\substack{\uparrow \\ \deg 1}}{\varepsilon})$

Def'n: The unipotent loop space is

$$\mathcal{L}^u(X) = \mathrm{Map}(BG_a, X)$$

Remark: $T_X[-1] = \mathrm{Map}(BG_a, X) \simeq \mathrm{Map}(S^1, X) = \mathcal{L}X$

For a derived scheme.

NOT true for derived stacks!

Thm (Ben-Zvi – Nadler)

$$D_{coh}(B \backslash G / B) \simeq \mathrm{Coh}_{[B \times B / G]} \left(\mathrm{St}^{\vee}_{G} / G \right)^{\$}_{loc}$$

follows from

Thm Let X be a "nice" derived stack with an underived loop space.

Then

$$D_{coh}(X) \simeq \mathrm{Coh}_{[X]} \left(\mathcal{L}^{\upsilon} X \right)^{\$}_{loc}$$

for $X = B \backslash G / B \simeq (B \times B) / G$